1. Introduction. In this note we produce an analog of the inverse spectral method (used to study, among other things, the continuous limit of the Toda lattice and the dispersionless limit of the KdV equation) for a class of "tri-diagonal" Toeplitz operators obtained from quantizing $\mathbb{S}^2$.

2. Toeplitz Quantization. We begin with some notation and a discussion of the quantization of $\mathbb{S}^2$.

Consider $\mathbb{C}\mathrm{P}^1$, the set of 1-dimensional complex subspaces of $\mathbb{C}^2$. Let $[z_0 : z_1]$ denote the subspace of $\mathbb{C}^2$ spanned by the vector $(z_0, z_1) \in \mathbb{C}^2 \backslash \{0\}$. These lead to coordinate charts U_j that cover $\mathbb{C}\mathrm{P}^1$ given by:

$$U_j = \{[z_0 : z_1]; \quad z_i \neq 0\}$$

with corresponding diffeomorphisms $\phi_0(z) = [1 : z]$ and $\phi_1(z) = [z : 1]$ that turns $\mathbb{C}\mathrm{P}^1$ into a complex manifold. Endow $\mathbb{C}\mathrm{P}^1$ with the Fubini-Study form given by:

$$\omega_{\mathrm{FS}} = i\partial\overline{\partial}\log(1 + |z|^2)$$

in complex coordinates. With this form $\mathbb{C}\mathrm{P}^1$ becomes a Kähler manifold.

Recall the *tautological line bundle*:

$$\mathcal{O}(-1) = \big\{([z_0 : z_1], (w_1, w_2)) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2; \quad (w_1, w_2) \in [z_0 : z_1]\big\}$$

with projection map $\pi([u], v) = [u]$ and consider the trivialization induced by the local coordinates for U_1:

$$\tau\colon \phi_1^{-1}(U_1) \times \mathbb{C} \to \phi_1^{-1}(U_1) \times \mathbb{C}^2; \qquad (z, w) \mapsto (z, w(z, 1)).$$

Define the following Hermitian form in this coordinate patch via the formula:

$$h_z(w, v) = (1 + |z|^2)w\overline{v}.$$

Then one can check that this Hermitian form extends to a Hermitian form on all of $\mathcal{O}(-1)$ and the curvature of the associated canoncial connection (the Chern connection) is:

$$\mathrm{curv}(\mathcal{O}(-1)) = -\partial\overline{\partial}\log(1 + |z|^2) = i\omega_{\mathrm{FS}}.$$

Therefore, if we take the dual of the tautological line bundle, $\mathcal{O}(1)$, we get a pre-quantum line bundle. Let $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ and let $\mathcal{H}_k$ be the space of holomorphic sections of $\mathcal{O}(k)$. Endow smooth sections of $\mathcal{O}(k)$ with the Hermitian form:

$$\langle f, g\rangle_k = \int_{\mathbb{C}\mathrm{P}^1} h_k(f, g)\,\mathrm{d}\lambda,$$

where h_k is the hermitian form on $\mathcal{O}(k)$ associated to h and λ is the Liouville measure associated to ω_{FS}. Let $L^2(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k))$ be the completion of $C^\infty(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k))$ with respect to the norm induced from $\langle\cdot,\cdot\rangle_k$. Define the *Szegö projector*:

$$\Pi_k\colon L^2(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k)) \to \mathcal{H}_k.$$

Given a function $H \in C^\infty(\mathbb{C}\mathrm{P}^1, \mathbb{R})$ we define its quantization to be:

$$T_H^{(k)}\colon L^2(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k)) \to L^2(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k)); \qquad f \mapsto \Pi_k(Hf).$$

In this setting a *Toeplitz operator* T corresponds to a sequence of operators $T^{(k)}\colon \mathcal{H}_k \to \mathcal{H}_k$ for $k = 1, 2, \ldots$ that admits an asymptotic expansion:

$$T^{(k)} \sim \sum_{j=0}^{\infty} k^{-j} T^{(k)}_{H_j},$$

where $\{H_j\}$ is a sequence of smooth functions and $\sim$ is meant in the operator norm.

2.1. Quantization of the principal bundle. Let Z denote the $\mathbb{S}^1$-bundle of elements in $\mathcal{O}(-1)$ of length one in each fiber. Using the local trivialization induced from the coordinate chart for U_1 we get a local trivialization of Z:

$$\widetilde{\tau}\colon \phi_1^{-1}(U_1) \times \mathbb{S}^1 \to \phi_1^{-1}(U_1) \times \mathbb{C}^2; \qquad (z, e^{i\theta}) \mapsto (z, \frac{e^{i\theta}}{(1+|z|^2)^{1/2}}(z,1)).$$

This trivialization essentially shows there is a bundle isomorphism $Z \cong \mathbb{S}^3 \subset \mathbb{C}^2$ (viewing $\mathbb{S}^3$ as an $\mathbb{S}^1$ bundle over $\mathbb{C}\mathrm{P}^1$ via the Hopf fibration). We will now identify Z with $\mathbb{S}^3$.

Let $C^\infty(\mathbb{S}^3)_k$ denote the space of k-equivariant (with respect to the $\mathbb{S}^1$-action) smooth $\mathbb{C}$-valued functions on $\mathbb{S}^3$. There is a natural isomorphism between $C^\infty(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k))$ and $C^\infty(\mathbb{S}^3)_k$. Using this identification, the Chern connection on $C^\infty(\mathbb{C}\mathrm{P}^1, \mathcal{O}(1))$ gives rise to a connection form on $C^\infty(\mathbb{S}^3)_k$. Denote this connection form by α. Then we get an invariant volume form on $\mathbb{S}^3$ from $(\frac{\alpha}{2\pi}) \wedge d\alpha$. It turns out that this volume form is a scalar multiple of the standard volume form on $\mathbb{S}^3$. For simplicity, we will assume that it is equal to the standard volume form. Thus, we have an isometry of Hilbert spaces:

$$L^2(\mathbb{C}\mathrm{P}^1, \mathcal{O}(k)) \cong L^2(\mathbb{S}^3, \mathbb{C})_k$$

We can identify any k-equivariant $\mathbb{C}$-valued smooth function as a k-homogenous smooth $\mathbb{C}$-valued function on $\mathbb{C}^2$. Similarly, given a smooth k-equivariant function on $\mathbb{S}^3$ we there is a corresponding unique smooth k-homogenous function on $\mathbb{C}^2$. Moreover, using the local section of $\mathbb{S}^3$ given by $z \mapsto \frac{1}{(1+|z|^2)^{1/2}}(z,1)$, we can identify an element of $C^\infty(\mathbb{S}^3, \mathbb{C})_k$ with an element of $C^\infty(\mathbb{C}, \mathbb{C})$ by:

$$C^\infty(\mathbb{S}^3, \mathbb{C})_k \to C^\infty(\mathbb{C}, \mathbb{C}); \qquad \frac{1}{(1+|z|^2)^{k/2}}\psi(z,1) \mapsto \psi(z).$$

Under the isometry of Hilbert spaces above, we get an associated Hermitian form:

$$\langle \psi(z), \phi(z) \rangle_k = \frac{1}{2} \int_{\mathbb{C}} \psi(z)\overline{\phi}(z) \frac{|dz \wedge d\bar{z}|}{(1+|z|^2)^{k+2}}.$$

The image of $\mathcal{H}_k$ under this isomorphism is given by holomorphic k-homogenous functions of two complex variables, which are just polynomials. Throughout the rest of this note we will make implicit reference to these identifications.

Lemma 2.1. *Under the previous identifications, the collection of polynomials* $\{e_\ell\}_{\ell=0}^{k}$*, where:*

$$e_\ell = \sqrt{\frac{(k+1)\binom{k}{\ell}}{\pi}}\, z^\ell,$$

form an orthonormal basis for $\mathcal{H}_k$. Consequently, the Schwartz Kernel for the Szegö projector is:

$$\Pi_k(z,w) = \frac{k+1}{\pi}(1+z\overline{w})^k.$$

2.2. Coordinates on $\mathbb{S}^2$. Recall that we can identify $\mathbb{C}\mathrm{P}^1$ as the quotient $\mathbb{S}^3/\mathbb{S}^1$. Then we get coordinates on $\mathbb{C}\mathrm{P}^1$ defined explicitly as:

$$x_1([z_1:z_2]) = \mathrm{Re}(z_1\overline{z_2}); \qquad x_2([z_1:z_2]) = \mathrm{Im}(z_1\overline{z_2});$$
$$x_3([z_1:z_2]) = \frac{1}{2}(|z_1|^2 - |z_2|^2).$$

This shows that $\mathbb{C}\mathrm{P}^1$ can be identified with the sphere of radius 1/2. Identifying $\mathbb{C}\mathrm{P}^1 = \mathbb{S}^3/\mathbb{S}^1$ via the local section $z \mapsto \frac{1}{(1+|z|^2)^{1/2}}(z,1)$ yields coordinate functions in the local trivialization:

$$x_1(z) = \frac{\mathrm{Re}(z)}{1+|z|^2}; \qquad x_2(z) = \frac{\mathrm{Im}(z)}{1+|z|^2}; \qquad x_3(z) = \frac{1}{2}\frac{|z|^2-1}{|z|^2+1}.$$

We will prefer to write the symbols of our Toeplitz operators in the usual (h,θ) coordinates on $\mathbb{S}^2$, where $h = x_3 + 1/2$.

3. Tri-diagonal Toeplitz operators and the spectral measure. Consider a smooth $\mathbb{R}$-valued function H on $\mathbb{S}^2$ given in (h,θ)-coordinates as $H(h,\theta) = a(h) + 2b(h)\cos\theta$. We get a corresponding Toeplitz operator by quantizing H:

$$T_H^{(k)} \colon \mathcal{H}_k \to \mathcal{H}_k; \qquad f \mapsto \Pi_k(Hf).$$

For a fixed k, $T_H^{(k)}$ is a self-adjoint linear operator on a finite dimensional vector space. Therefore, we can orthogonally diagonalize $T_H^{(k)}$. Moreover, with respect to the orthonormal basis for $\mathcal{H}_k$ given in Section 2, each $T_H^{(k)}$ is tri-diagonal. Following [Dei99, Chapter 2] (and assuming that our H is chosen so that the off-diagonal terms of $T_H^{(k)}$ are all positive), we can orthogonally diagonalize $T_H^{(k)}$ and let $\lambda_i^{(k)}$ denote the eigenvalues (which are necessarily simple) and let $w_i^{(k)}$ denote the first component of a normalized eigenvector corresponding to eigenvalue $\lambda_i^{(k)}$ (which is necessarily non-zero). For each k form the so-called *spectral measure* of $T_H^{(k)}$:

$$d\mu_k(s) = \sum_{i=0}^{k} (w_i^{(k)})^2 \delta(s - \lambda_i^{(k)}).$$

The moments of these measures are given by the formulas:

$$\int_{\mathbb{R}} s^n \, d\mu_k(s) = \langle T_H^n e_0, e_0 \rangle_k.$$

For a *fixed* k the map $T_H^{(k)} \mapsto \mu_k$ is injective. We will prove that if a and $\frac{1}{\sqrt{h(1-h)}}b$ are real analytic, then H can be recovered from knowledge of the first two moments of every μ_k.

We start by analyzing the tri-diagonal Toeplitz operator H using the identifications from Section 2.

Lemma 3.1. *Let $H(h,\theta) = a(h) + 2b(h)\cos\theta$ be a Hamiltonian on $\mathbb{S}^2$. Then under the identifications from Section 2 the quantization of H is the Toeplitz operator $(T^{(k)})$ where:*

(3.1)
$$T^{(k)}\colon \mathcal{H}_k \to \mathcal{H}_k; \quad \psi(z) \mapsto \Pi_k \left[a\left(\frac{|z|^2}{|z|^2+1}\right)\psi(z) + \frac{2\mathrm{Re}(z)}{1+|z|^2}\beta\left(\frac{|z|^2}{|z|^2+1}\right)\psi(z)\right],$$

where $\beta(h) = \frac{1}{\sqrt{h(1-h)}}b(h)$ (which is a smooth function on $\mathbb{S}^2$).

Proof. The z-coordinate expression of h is $\frac{|z|^2}{|z|^2+1}$ which explains the first term in (3.1). For the second term, put $\beta(h) = \frac{1}{\sqrt{h(1-h)}}b(h)$. The fact that $\beta(h)$ is a smooth function on the sphere follows from Lemma 3.6 of [BGPU03]. Recall the identity for cylindrical coordinates on the sphere: $2x_1 = 2\cos\theta\sqrt{h(1-h)}$. Then from the definition of $\beta(h)$ we have:

$$2x_1\beta(h) = 2\cos\theta\sqrt{h(1-h)}\beta(h) = 2b(h)\cos\theta.$$

The claim follows from the z-coordinate expression for x_1. □

Proposition 3.2. *Assume that $\beta(0) > 0$. Then the first two moments of μ_k yield asymptotic expansions (in k) that determine the Taylor series coefficients of a and β at zero. Consequently, if a and β are real analytic, one can determine H from the first two moments of every μ_k.*

Proof. We break the proof up into several claims.

Claim 1. The Taylor series coefficients of a centered at zero are determined by the asymptotics of the first moments.

Fix k. Then the first moment of $T_H^{(k)}$ is: $\langle T_H^{(k)} e_0, e_0\rangle_k$. From Lemma 3.1:

(3.2)
$$\langle T_H^{(k)} e_0, e_0\rangle_k = \frac{(k+1)}{2\pi}\int_{\mathbb{C}} \left[a\left(\frac{|z|^2}{|z|^2+1}\right) + \frac{2\mathrm{Re}(z)}{1+|z|^2}\beta\left(\frac{|z|^2}{|z|^2+1}\right)\right] \frac{|dz\wedge d\bar{z}|}{(1+|z|^2)^{k+2}}.$$

The integral of the second term is zero by symmetry. Therefore:

$$\begin{aligned}\langle T_H^{(k)} e_0, e_0\rangle_k &= \frac{k+1}{2\pi}\int_{\mathbb{C}} a\left(\frac{|z|^2}{|z|^2+1}\right)\frac{|dz\wedge d\bar{z}|}{(1+|z|^2)^{k+2}} \\ &= \int_0^\infty a\left(\frac{r^2}{1+r^2}\right)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr,\end{aligned}$$

where in the second line we changed to polar coordinates and integrated away the θ-dependence. A tedious calculation reveals that the first $2n$ Taylor series coefficients of $a\left(\frac{r^2}{1+r^2}\right)$ at $r=0$ determine the first n Taylor series coefficients of $a(s)$ at $s=0$. Therefore, by Lemma 3.3 we have:

$$\int_0^\infty a\left(\frac{r^2}{1+r^2}\right)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr \sim \sum_{n=0}^{\infty}\frac{\widetilde{a}^{(n)}(0)(k-\frac{n}{2})!\frac{n}{2}!}{n!k!}, \tag{3.3}$$

where $\widetilde{a}(r) = a\left(\frac{r^2}{1+r^2}\right)$. The claim follows.

Claim 2. For every k the vector e_0 is an eigenvector of $T_a^{(k)}$. Consequently, the asymptotic expansion for the corresponding eigenvalue of e_0 is given by (3.3).

To prove this claim we compute (using Lemma 2.1):

$$\begin{aligned} T_a^{(k)} e_0 &= \frac{1}{2}\int_{\mathbb{C}} \Pi_k(z,w)a(h)e_0 \frac{|dw \wedge d\overline{w}|}{(1+|w|^2)^{k+2}} \\ &= \frac{1}{2}\int_{\mathbb{C}} \frac{k+1}{\pi}(1+z\overline{w})^k a\left(\frac{|w|^2}{1+|w|^2}\right) e_0 \frac{|dw \wedge d\overline{w}|}{(1+|w|^2)^{k+2}}. \end{aligned} \tag{3.4}$$

Using the last line of (3.4) to regard $T_a^{(k)} e_0$ as a function of z, we see that the transformation $z \mapsto e^{i\theta} z$ is equivalent to the change of integration variables $w \mapsto e^{-i\theta} w$. However, the integral in the last line of (3.4) is obviously invariant under this change of variables. Therefore, $T_a^{(k)} e_0$ is not a function of z. The claim follows.

Claim 3. For every k we have: $\langle T_a^{(k)} \circ T_{2b\cos\theta}^{(k)} e_0, e_0\rangle_k = \langle T_{2b\cos\theta}^{(k)} \circ T_a^{(k)} e_0, e_0\rangle_k = 0.$

We compute:

$$\begin{aligned} \langle T_a^{(k)} \circ T_{2b\cos\theta}^{(k)} e_0, e_0\rangle_k &= \langle T_{2b\cos\theta}^{(k)} e_0, T_a^{(k)} e_0\rangle_k \\ &= \lambda \langle T_{2b\cos\theta}^{(k)} e_0, e_0\rangle_k \\ &= 0. \end{aligned}$$

The first equality comes from the fact that $T_a^{(k)}$ is self-adjoint. The second equality comes from Claim 2, where λ is the eigenvalue of e_0. The third equality comes from (3.3). A similar computation establishes the claim for the other ordering of the operators.

The proposition follows from:

Claim 4. The Taylor series coefficients of β centered at zero are determined by the asymptotics of the second moments.

By Claim 3,

$$\begin{aligned} \langle T_H^{(k)} \circ T_H^{(k)} e_0, e_0\rangle_k &= \langle T_a^{(k)} \circ T_a^{(k)} e_0, e_0\rangle_k + \langle T_{2b\cos\theta}^{(k)} \circ T_{2b\cos\theta}^{(k)} e_0, e_0\rangle_k \\ &= \|T_a^{(k)} e_0\|_k^2 + \|T_{2b\cos\theta}^{(k)} e_0\|_k^2. \end{aligned}$$

Since the asymptotic expansion of the first term on the right hand side is determined from the asymptotic expansion of the first moments, then it suffices to show that the Taylor series coefficients of β at zero are determined from the second term.

We compute:

$$\begin{aligned} \|T_{2b\cos\theta}^{(k)} e_0\|_k^2 &= \frac{k+1}{2\pi}\int_{\mathbb{C}} \frac{4\mathrm{Re}(z)^2}{(1+|z|^2)^2}\beta^2\left(\frac{|z|^2}{1+|z|^2}\right)\frac{|dz \wedge d\overline{z}|}{(1+|z|^2)^{k+2}} \\ &= \frac{k+1}{2\pi}\int_0^\infty \frac{4\pi r^2}{(1+r^2)^2}\beta^2\left(\frac{r^2}{1+r^2}\right)\frac{rdr}{(1+r^2)^{k+2}} \\ &= \int_0^\infty \frac{r^2}{(1+r^2)^2}\beta^2\left(\frac{r^2}{1+r^2}\right)\frac{2r(k+1)dr}{(1+r^2)^{k+2}} \\ &\sim \sum_{n=0}^\infty \frac{g^{(n)}(0)(k-\frac{n}{2})!\frac{n}{2}!}{n!k!}, \end{aligned}$$

where in the first equality we changed to polar coordinates and integrated away the θ-variables. The asymptotics in the last line come from applying Lemma 3.3, with

$$g(r) = \frac{r^2}{(1+r^2)^2}\beta^2\left(\frac{r^2}{1+r^2}\right).$$

To see that the Taylor series coefficients of g determine the coefficients of β, first notice that if $\beta(0) > 0$ then the derivatives of $\beta^2(s)$ evaluated at $s = 0$ determine the derivatives of $\beta(s)$ at $s = 0$. Second, as observed in Claim 1, the first $2n$ Taylor series coefficients of $\beta^2\left(\frac{r^2}{1+r^2}\right)$ centered at $r = 0$ determine the first n Taylor series coefficients of $\beta^2(s)$ at $s = 0$. Finally, notice that the first $n+2$ Taylor series coefficients of $\frac{r^2}{(1+r^2)^2}\beta^2\left(\frac{r^2}{1+r^2}\right)$ at $r = 0$ determine the first n Taylor series coefficients of $\beta^2\left(\frac{r^2}{1+r^2}\right)$ at $r = 0$. This completes the proof of the claim. □

Lemma 3.3. *Let $f(r)\colon \mathbb{R} \to \mathbb{R}$ be smooth and bounded. Then:*

$$\int_0^\infty f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr \sim \sum_{n=0}^\infty \frac{f^{(n)}(0)(k-\frac{n}{2})!\frac{n}{2}!}{n!k!},$$

as $k \to \infty$.

Proof. The proof is very similar to *Watson's Lemma* (see for instance [Mil06, Section 2.2]). Let $s > 0$ and break up the integral:

$$\int_0^\infty f(r)\frac{2r(k+1)}{(1+r)^{k+2}}\,dr = \int_0^s f(r)\frac{2r(k+1)}{(1+r)^{k+2}}\,dr + \int_s^\infty f(r)\frac{2r(k+1)}{(1+r)^{k+2}}\,dr.$$

Analyzing the second term, we get:

$$\begin{aligned}\left|\int_s^\infty f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr\right| &\le \int_s^\infty \left|f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\right|\,dr \\ &\le \frac{k+1}{(1+s^2)^k}\int_s^\infty \left|f(r)\frac{2r}{(1+r^2)^2}\right|\,dr.\end{aligned}$$

Therefore, $\int_s^\infty f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr = o(k^{-n})$ for every n.

Next, we analyze the integral from 0 to s. Fix N, and consider the Taylor series expansion (with remainder) of f centered at zero:

$$f(r) = \sum_{n=0}^N \frac{f^{(n)}(0)}{n!}r^n + R_n(r),$$

where $|R_n(r)| \le \sup_{\tau\in[0,s]} |f^{(N+1)}(\tau)|\frac{r^{N+1}}{(N+1)!}$. Replacing $f(r)$ in the integrand with its Taylor series expansion yields:

$$\int_0^s f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr = \int_0^s \sum_{n=0}^N \frac{f^{(n)}(0)}{n!}\frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr + \int_0^s R_N(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr.$$

Note that:

$$\int_0^s \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr = \int_0^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr - \int_s^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr$$
$$= \frac{(1+k)\Gamma(1+k-\frac{n}{2})\Gamma(1+\frac{n}{2})}{\Gamma(2+k)} - \int_s^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr$$
$$= \frac{(k-\frac{n}{2})!\frac{n}{2}!}{k!} - \int_s^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr,$$

provided $2k-n > -2$. Analyzing the remaining integral from s to ∞ using Cauchy-Schwarz, we get:

$$\int_s^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr \leq \sqrt{\int_s^\infty \frac{(k+1)r}{(1+r^2)^{k+2}}\,dr}\sqrt{\int_s^\infty \frac{2r^{2n+1}(k+1)}{(1+r^2)^{k+2}}\,dr}$$
$$\leq \frac{(k+1)^{1/2}}{(1+s^2)^{k/2}}\sqrt{\int_s^\infty \frac{r}{(1+r^2)^2}\,dr}\sqrt{\int_s^\infty \frac{2r^{2N+1}(k+1)}{(1+r^2)^{k+2}}\,dr}.$$

Provided $2k-2N > -2$, we conclude that $\int_s^\infty \frac{2r^{n+1}(k+1)}{(1+r^2)^{k+2}}\,dr = o(k^{-\ell})$ for every $\ell > 0$. Therefore, we obtain the asymptotic expansion:

$$\int_0^s f(r)\frac{2r(k+1)}{(1+r^2)^{k+2}}\,dr \sim \sum_{n=0}^\infty \frac{f^{(n)}(0)(k-\frac{n}{2})!\frac{n}{2}!}{n!k!},$$

and the claim follows. □

4. Time evolution of the spectral measure. The goal of this section is to sketch an idea to evolve the spectral measure according to the Toda lattice time evolution and relate that back to the time evolution of H according to the Toda PDE. Let L_k be the matrix corresponding to the operator $T_H^{(k)}$ with respect to the basis $\{e_\ell\}_{\ell=0}^k$. Evolve L_k according to the Toda equations, given by:

$$\dot{L}_k = [B(L_k), L_k], \qquad B(L_k) = L_{k,-} - (L_{k,-})^\top, \tag{4.1}$$

where $L_{k,-}$ is the strictly lower triangular part of L_k. To solve the Toda equation, write:

$$e^{tL_k} = Q(t)R(t),$$

where $Q(t)$ is orthogonal and $R(t)$ is upper triangular. Then the solution of (4.1) is:

$$Q^\top(t)L_kQ(t).$$

We are interested in the time evolution of the spectral measure, which is given by:

$$d\mu(t) = \langle Qe_0, L_kQe_0\rangle, \tag{4.2}$$

where we have suppressed the dependence of Q on t.

Possible lines of inquiry:

(1) The measures $d\mu(t)$ exists for all time, however, the solutions to the corresponding Toda PDE do not necessarily. Is there some way of detecting shocks in the Toda PDE using $d\mu(t)$?

(2) Investigate the time evolution of these measures numerically.

(3) Develop an asymptotic expansion for $d\mu(t)$ and compare it to the solution of the Toda PDE. (This is really wishful thinking.)

Example.

References.

[BGPU03] A. Bloch, F. Golse, T. Paul, and A. Uribe. Dispersionless Toda and Toeplitz operators. *Duke Math. J.*, 117(1):157–196, 2003.

[Dei99] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[Mil06] Peter D. Miller. *Applied asymptotic analysis*, volume 75 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.